On the Flow Stability of
Finite Element Approximations
of the Navier-Stokes Equations

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1. Introduction

Flow stability in fluid mechanics is an invaluable concept in order to predict whether a flow is sensitive to small perturbations. In principle, the perturbations either decay or grow with time. When they decay, the perturbed flow returns to its original state and the flow is said to be stable. When the perturbations grow, the perturbed flow evolve to another state of motion and the flow is said to be unstable. In particular, the loss of stability is held responsible for the transition phenomena from a laminar to a turbulent state. One is then interested in understanding the complex mechanisms involved in transition in order to design adequate devices aimed at controlling the flow turbulence, either to sustain a turbulent motion (combustion) or to prevent it from occurring (drag reduction).

Flow stability and the associated transition phenomena have been investigated using two approaches, namely the hydrodynamic stability theory and the dynamical system theory. We will present a brief account on both subjects in order to introduce the relevant terminology and concepts which will be useful for the interpretation of our numerical experiments.

The objectives here are primarily to analyse the flow stability of finite element approximations of the Navier-Stokes equations, and in particular, to investigate whether the stability properties of numerical flows are modified due to the discretization in space. With this scope in mind, we chose to perform numerical simulations of the channel flow past a cylinder and to compare the dynamical behavior of our finite element solutions with respect to various mesh sizes. The numerical experiments are described and analysed in the last two sections of this report, followed by a conclusion.
2. Preliminaries

Let $\Omega$ denote an open bounded domain in $\mathbb{R}^d$, $d = 2$ or 3, with boundary $\partial \Omega$. The flow of a viscous incompressible fluid in $\Omega$ is modeled by the Navier-Stokes equations,

$$\partial_t u + (u \cdot \nabla) u - Re^{-1} \Delta u + \nabla p = f$$
$$\nabla \cdot u = 0$$

in $\Omega \times (0, T)$

with prescribed boundary condition $u(\mathbf{x}, t) = 0$, for all $\mathbf{x} \in \partial \Omega$ and $t \in (0, T)$, and initial condition $u(\mathbf{x}, 0) = u_0(\mathbf{x})$, for all $\mathbf{x} \in \Omega$. Here $u$ and $p$ respectively denote the velocity vector and the pressure, $Re$ is the Reynolds number, and $f = f(\mathbf{x}, t)$ is a prescribed body force. Let $V$ and $Q$ be the spaces of trial velocities and pressures respectively, $V = [H_0^1(\Omega)]^d$, and $Q = \{q \in L^2(\Omega) : \int_{\Omega} q \, dx = 0\}$, and let $\|\cdot\|_0$, $\|\cdot\|_1$ denote the usual $[L^2(\Omega)]^d$-norm and $[H_0^1(\Omega)]^d$-norm for elements in $V$ and $\|\cdot\|_0$ the $L^2(\Omega)$-norm for elements in $Q$.

A variational form of the Navier-Stokes problem reads: for $f$ and $u_0$ given, find $u \in L^2(0, T; V)$ and $p \in L^2(0, T; Q)$ such that, for almost all $t \in (0, T)$:

$$\partial_t u + c(u, u, v) + Re^{-1} a(u, v) + b(v, p) = F(v), \quad \forall v \in V,$$
$$b(u, q) = 0, \quad \forall q \in Q;$$
$$u = u_0, \quad \text{at} \ t = 0.$$  \hspace{1cm} (2.2)

where the forms $a$, $b$, $c$ and $F$ are defined for $u, v, w \in V$ and for $q \in Q$ as:

$$a(u, v) = \int_{\Omega} \nabla u : \nabla v \, dx,$$
$$b(v, q) = -\int_{\Omega} q \nabla \cdot v \, dx,$$
$$c(u, v, w) = \int_{\Omega} (u \cdot \nabla) v \cdot w \, dx,$$
$$F(v) = \int_{\Omega} f \cdot v \, dx.$$  

We assume that there exists a unique solution $(u, p) \in L^2(0, T; V) \times L^2(0, T; Q)$ of Problem (2.2), which can be proved in dimension $d = 2$ under mild conditions on $f$ and $u_0$, but which has not been proved or disproved in dimension $d = 3$ (see, e.g., Temam [51, 52, 50], and Constantin and Foias [7]).

The above problem is approximated by the Galerkin method using $h$-$p$ finite element spaces $V^h \subset V$ and $Q^h \subset Q$ [9]. It is now well-known [23] that the spaces $V^h$ and $Q^h$ must be chosen in order to satisfy the discrete version of the LBB inf-sup condition. This means that $V^h$ and $Q^h$ are constructed so that there exists a constant $\beta_h > 0$ such that:

$$\sup_{v \in V^h \setminus \{0\}} \frac{|b(v, q)|}{\|v\|_1} \geq \beta_h \|q\|_0, \quad \forall q \in Q^h.$$  \hspace{1cm} (2.3)
In the case of hierarchical shape functions, we shall use the following rule inspired by the work of Suri et al. [46, 6] on locking-free \( h-p \) elements: the spectral order of the shape functions for the pressure variable is chosen at least one less for the edge functions and at least two less for the interior functions than the spectral order of the shape functions for the velocity variable. The finite element solution \((u_h, p_h) \in L^2(0, T; \mathbb{V}^h) \times L^2(0, T; Q^h)\) is then governed by the following semi-discrete system of equations, for almost every \( t \in (0, T) \):

\[
(\partial_t u_h, v) + c(u_h, u_h, v) + Re^{-1} a(u_h, v) + b(v, p_h) = F(v), \quad \forall v \in \mathbb{V}^h,
\]

\[
b(u_h, q) = 0, \quad \forall q \in Q^h,
\]

\[
u_h = u_0^h, \quad \text{at} \ t = 0,
\]

where \( u_0^h \) is an appropriate projection of \( u^0 \) on the finite element space \( \mathbb{V}^h \).

From a practical point of view, the semi-discrete problem above is never solved as is. Indeed, the differential equations are also discretized in time to provide a fully discrete problem. In the following, the Navier-Stokes equations are discretized using the Adams-Bashforth Crank-Nicholson scheme, which is conditionally stable. Let \( \Delta t \) denote the timestep. Then, given some initial conditions \( u_0^h \) and \( \mathbf{u}_1^h \), the approximate solution \((u_h^n, p_h^n) \in \mathbf{V}^h \times Q^h\) at the discrete times \( t^n = n \Delta t, n = 2, 3, \ldots \) is advanced in time by solving the system of equations

\[
\frac{1}{\Delta t}(u_h^n, v) + \frac{1}{2} Re^{-1} a(u_h^n, v) + b(v, p_h^n) = \mathcal{F}_h(v), \quad \forall v \in \mathbb{V}^h
\]

\[
b(u_h^n, q) = 0, \quad \forall q \in Q^h
\]

where \( \mathcal{F}_h \) is a functional of \( v \) which depends on the solutions at the previous timesteps, i.e.

\[
\mathcal{F}_h(v) = \frac{1}{2} F^n(v) + \frac{1}{2} F^{n-1}(v) + \frac{1}{\Delta t}(u_h^{n-1}, v) - \frac{1}{2} Re^{-1} a(u_h^{n-1}, v)
\]

\[
- \frac{3}{2} c(u_h^{n-1}, u_h^{n-1}, v) + \frac{1}{2} c(u_h^{n-2}, u_h^{n-2}, v). \tag{2.6}
\]

The ABCN scheme is clearly implicit in the linear terms and explicit for the nonlinear convective terms.

### 3. Hydrodynamics stability theory

For illustrative purpose, we consider an incompressible fluid flowing between two infinite parallel planes located at \( y = \pm 1 \) in the reference frame \((0, x, y, z)\). A possible state for this flow is the plane \textit{Poiseuille} flow, whose velocity profile \( \mathbf{U} = (U, V, W) \) is given, in dimensionless form, by

\[
U = U(x, t) = (1 - y^2), \quad V = V(x, t) = 0, \quad W = W(x, t) = 0. \tag{3.1}
\]
The plane Poiseuille flow is independent of the Reynolds number $Re$ and thus represents a possible steady-state for all values of $Re$. Mathematically speaking, if one is able to attain the Poiseuille flow (3.1) at the initial time $t_0$ between two infinite parallel planes, and if absolutely no perturbations are introduced to the system, then the flow profile (3.1) will be maintained for all subsequent times $t \geq t_0$ and for any value of $Re$. In laboratory experiments, however, the Poiseuille flow profile is observed at low Reynolds numbers only; when $Re$ is increased past a critical value, this flow naturally evolves into a "turbulent" time-dependent state as it becomes unstable when subjected to small disturbances, always present in actual experiments. One of the primary roles of hydrodynamics stability theory is to predict the value of the critical Reynolds number at which the flow starts to become unstable.

Let $(U, P)$ define a steady-state basic flow in a region $\Omega$ (for example the Poiseuille flow), $U$ being the velocity vector and $P$ the pressure. The starting point in stability analysis is to consider a perturbation to the basic flow $(U, P)$. Let us assume here that a perturbation in the velocity profile $U$ is introduced at time $t = 0$. This results in a perturbed flow $(u', p')$, which satisfies the Navier-Stokes equations as well, and the evolution of the perturbation $(u', p')$

$$u'(x, t) = u(x, t) - U(x), \quad p'(x, t) = p(x, t) - P(x), \quad (3.2)$$

is easily shown to be governed by the nonlinear system of equations:

$$\partial_t u' + u' \cdot \nabla U + U \cdot \nabla u' + u' \cdot \nabla u' - Re^{-1} \Delta u' + \nabla p' = 0 \quad (3.3)$$

$$\nabla \cdot u' = 0 \quad (3.4)$$

and to be subject to the boundary condition $u' = 0$ on $\partial \Omega$ and to the initial condition $u'(0) = u' - U$ at $t = 0$. Stability analysis consists then in the study of the evolution and effects of the perturbation $(u', p')$. The two classical approaches to date are the linear theory or the energy method.

3.1. Linear theory

There exists an extensive literature on linear theory, for example [31, 41, 3, 25, 11, 19], so we briefly outline the methodology here. In linear theory, one assumes that the fluctuating velocity $u'$ is infinitesimal, in the sense that the quadratic term $u' \cdot \nabla u'$ becomes negligible with respect to the linear terms. Therefore, the "momentum" equation (3.3) simplifies to:

$$\partial_t u' + u' \cdot \nabla U + U \cdot \nabla u' - Re^{-1} \Delta u' + \nabla p' = 0. \quad (3.5)$$

When the basic flow is parallel, i.e. $U = (U(y), 0, 0)$, such as the plane Poiseuille flow, all the coefficients in the linear equation (3.5) become independent of $x$, $z$ and $t$. Moreover, if the perturbed flow $(u, p)$ is restricted to be two-dimensional,
then it seems reasonable to consider a class of perturbations \( \{ (u', v', p')_{a, c} \} \) of the wave type:

\[
\begin{align*}
  u'(x, y, t) &= \hat{u}(y) \exp \{ i \alpha (x - ct) \} \\
  v'(x, y, t) &= \hat{v}(y) \exp \{ i \alpha (x - ct) \} \\
  p'(x, y, t) &= \hat{p}(y) \exp \{ i \alpha (x - ct) \}
\end{align*}
\]  

(3.6)

where \( \alpha \) is a real number such that \( \lambda = 2\pi /\alpha \) is the wavelength, \( c \) is a complex number, \( c = c_r + ic_i \), with \( c_i \) measuring the degree of amplification/damping and \( c_r \) being the celerity of the wave.

Introducing (3.6) in equations (3.5) and (3.4), one obtains the so-called Orr-Sommerfeld equation in the variable \( \hat{v}(y) \):

\[
(U - c) \left( \frac{\partial^2 \hat{v}}{\partial y^2} - \alpha^2 \hat{v} \right) - \frac{\partial^2 U}{\partial y^2} \hat{v} = - \frac{i}{\alpha Re} \left( \frac{\partial^4 \hat{v}}{\partial y^4} - 2\alpha \frac{\partial^2 \hat{v}}{\partial y^2} + \alpha^2 \hat{v} \right). 
\]  

(3.7)

Equation (3.7) was derived in 1907 and 1908 by Orr and Sommerfeld respectively. It determines, together with appropriate boundary conditions, an eigenvalue problem used to establish the stability of parallel flows with respect to the class of infinitesimal perturbations (3.6). Indeed, by fixing the values of \( Re \) and \( \alpha \), this problem provides some eigenvalues \( c = c_r + ic_i \) with associated eigenfunctions \( \hat{v}(y) \). The sign of \( c_i \) determines whether the perturbation \( (u', v', p')_{a, c} \) is amplified \( (c_i > 0) \) or damped \( (c_i < 0) \) with time. Subsequently, the flow \( (U, P) \), at a given Reynolds number \( Re \), is said to be either stable, if all the perturbations \( (u', v', p')_{a, c} \) are damped, or unstable, if there exists at least one perturbation which grows with time.

The eigenvalue problem (3.7) was eventually solved by Tollmien in 1929 and Schlichting in 1933 for boundary layer flows using an approximation of the Blasius profile. They established the existence of at least one unstable mode for \( Re \) high enough, commonly named the Tollmien-Schlichting (TS) wave. The existence of TS waves was later confirmed experimentally by Schubauer and Skramstad [42] in 1947, who observed a periodic flow by artificially exciting the laminar flow over a flat plate. Since then, many efforts have been devoted to understand the instability mechanisms for the primary (steady-state) and also the secondary (periodic) flow. Laboratory experiments, see for example [28, 54, 26] in the case of boundary layer flows or [36] in the case of Poiseuille flows, were performed, accompanied by analytical analyses [31, 48, 49, 1, 24] and numerical experiments [37, 38, 54, 27, 14, 15, 29]. These studies helped to partially describe the transition phenomena from laminar to turbulent flows, but failed to provide the complete picture. Unfortunately, this approach is limited to the stability analyses of simple parallel flows like the Couette, plane Poiseuille and Hagen-Poiseuille flows, or nearly parallel flows, such as the Blasius flow over a flat plate. Alternatively, the energy method is a more general approach.
3.2. Energy method

We first reconsider the weak form of the Navier-Stokes equations by looking for solutions in the closed subspace $\mathbf{J}$ of $\mathbf{V}$ whose functions are divergence-free, i.e.

$$\mathbf{J} = \{ \mathbf{v} \in \mathbf{V}; \nabla \cdot \mathbf{v} = 0 \}.$$ 

Then, for $\mathbf{f}$ and $\mathbf{u}^0$ given, one wants to find solutions of the Navier-Stokes problem in the space $L^2(0, T; \mathbf{J})$ satisfying, for all $t \in (0, T)$:

$$\begin{align*}
(\partial_t \mathbf{u}, \mathbf{v}) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}) + Re^{-1} a(\mathbf{u}, \mathbf{v}) &= F(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{J}, \quad (3.8) \\
\mathbf{u} &= \mathbf{u}^0, \quad \text{at } t = 0. \quad (3.9)
\end{align*}$$

Let $\mathbf{U} \in \mathbf{J}$ denote a steady-state basic flow, solution of the Navier-Stokes equations. A perturbation to the basic flow will be denoted by $\mathbf{u}' \in L^2(0, T; \mathbf{J})$. By class of perturbations to the flow $\mathbf{U}$, we mean the subset $\{ \mathbf{u}' \}$ of functions $\mathbf{u}' \in L^2(0, T; \mathbf{J})$ induced by a set of perturbed initial states $\{ \mathbf{u}^0 \}$ with $\mathbf{u}'(0) = \mathbf{u}^0 - \mathbf{U}$. In the following, we consider the space $L^2(0, T; \mathbf{J})$ as the class of perturbations. Replacing $\mathbf{u}$ by $(\mathbf{U} + \mathbf{u}')$ in (3.8), we derive the evolution equation governing the perturbation $\mathbf{u}'$, which reads, for almost all $t \in (0, T)$:

$$\begin{align*}
(\partial_t \mathbf{u}', \mathbf{v}) + c(\mathbf{U}, \mathbf{u}', \mathbf{v}) + c(\mathbf{u}', \mathbf{U}, \mathbf{v}) \\
+ c(\mathbf{u}', \mathbf{u}'', \mathbf{v}) + Re^{-1} a(\mathbf{u}', \mathbf{v}) &= 0, \quad \forall \mathbf{v} \in \mathbf{J}, \quad (3.10)
\end{align*}$$

as well as the initial condition $\mathbf{u}'(0) = \mathbf{u}^0 - \mathbf{U}$ at $t = 0$.

**Definition 3.1 (Stability in the Energy)** A flow $\mathbf{U}$ is said to be stable in the energy with respect to a class of perturbations $\{ \mathbf{u}' \}$ if the $L^2(\Omega)$ norm of all the perturbations in $\{ \mathbf{u}' \}$ eventually decays with time, i.e.

$$\forall \epsilon > 0, \exists \tau = \tau(\epsilon) > 0 \text{ such that } \| \mathbf{u}'(t) \|_0 < \epsilon, \quad \forall t > \tau. \quad (3.11)$$

Otherwise the flow is said to be unstable.

A sufficient condition to assess the stability properties of a basic flow $\mathbf{U}$ states that all perturbations in $L^2(0, T; \mathbf{J})$ should satisfy:

$$\frac{1}{2} \frac{d}{dt} \| \mathbf{u}' \|_0^2 < 0, \quad (3.12)$$

where the quantity $1/2 \| \mathbf{u}' \|_0^2$ represents the kinetic energy of the perturbation $\mathbf{u}'$ assuming a unity fluid density. If such a condition is satisfied, the flow $\mathbf{U}$ is said to be stable in the energy sense. The evolution of the kinetic energy of $\mathbf{u}'$ is obtained by substituting $\mathbf{u}'$ for $\mathbf{v}$ in (3.10):

$$\begin{align*}
\frac{1}{2} \frac{d}{dt} \| \mathbf{u}' \|_0^2 &= -c(\mathbf{U}, \mathbf{u}', \mathbf{u}') - c(\mathbf{u}', \mathbf{U}, \mathbf{u}') - c(\mathbf{u}', \mathbf{u}', \mathbf{u}') - Re^{-1} a(\mathbf{u}', \mathbf{u}') \\
&= -c(\mathbf{u}', \mathbf{U}, \mathbf{u}') - Re^{-1} a(\mathbf{u}', \mathbf{u}') \\
&= -c(\mathbf{u}', \mathbf{U}, \mathbf{u}') - Re^{-1} |\mathbf{u}'|^2.
\end{align*}$$
Therefore the flow $U$ is stable if for all $t > 0$ and all $u'(t) \in J$, assuming $u'(t) \neq 0$:

$$-Re \frac{c(u', U, u')}{|u'|^2} < 1$$

The stability problem is then equivalent to finding the supremum $\mu$, dependent on the Reynolds number $Re$, of the variational problem:

$$\mu = -Re \sup_{v \in J \setminus \{0\}} \left[ \frac{c(v, U, v)}{|v|^2} \right] = -Re \sup_{|v| = 1} \frac{c(v, U, v)}{|v|^2},$$

and establishing the stability of $U$ by verifying that $\mu < 1$.

The variational problem above is recast as an eigenvalue problem using classical tools of variational theory. Let $\lambda$ denote a Lagrange multiplier in $\mathbb{R}$ and $F$ define the bilinear functional $F : J \times \mathbb{R} \rightarrow \mathbb{R}$ such that:

$$F(v, \lambda) = -c(v, U, v) - \lambda Re^{-1}[a(v, v) - 1].$$

Necessary conditions for $F$ to have non-trivial stationary points $\omega \in J$ are that $\delta v F(\omega, \lambda) = 0$. Consequently, the extrema of the functional $F$ are defined by the eigenpairs $(\omega, \lambda) \in J \times \mathbb{R}$ which satisfy the following eigenvalue problem:

$$\frac{1}{2}[c(\omega, U, v) + c(v, U, \omega)] = -\lambda Re^{-1} a(\omega, v), \quad \forall v \in J.$$  

This eigenvalue problem is symmetric, and so provides a spectrum of real eigenvalues. Serrin [43] related the spectrum of eigenvalues to the extremum $\mu$ of the variational problem (3.14):

**Theorem 3.1 (Serrin)** If the variational problem (3.14) admits a solution, i.e. if there exists $\omega \in J$ such that:

$$\mu = -Re \frac{c(\omega, U, \omega)}{|\omega|^2}$$

then $\mu$ is equal to the largest eigenvalue of the eigenvalue problem (3.16) and the flow $U$ is said to be stable if $\mu < 1$.

The proof is found in Georgescu [19]. Serrin’s theorem, however, left open the issue of existence of a maximum of (3.14). We note that this problem has been proved by Rionero (see [17]).

In summary, the energy method reduces the stability analysis to solving an eigenvalue problem with real eigenvalues $\{\lambda_i\}$. The flow $U$ is said to be stable if $\max \{\lambda_i\} < 1$, and unstable if there exists at least one eigenvalue $\lambda_i \geq 1$. 

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If such an unstable perturbation \( u' \) exists, then the flow \( u = U + u' \) would eventually evolve into a state different from \( U \). We remark that \( U \) was assumed to represent a steady-state, but the energy method can be extended to unsteady flows \( U(t) \) [17]. We shall apply the energy method to study the stability of finite element approximations of the Navier-Stokes equations in a later section.

4. Dynamical system theory

The literature on *Dynamical System Theory* is rather broad, so we refer to the selected books by Sparrow [45], Bergé, Pomeau and Vidal [2], Seydel [44], McCauley [34], Verhulst [53], Stuart and Humphries [47] for general systems, and to the publications by Ruelle and Takens [40], Eckmann and Ruelle [12], Constantin, Foias and Temam [8], Doering and Gibbon [10] for analyses applied to the mechanics of fluids.

4.1. Dynamical systems

We begin by the definition of dynamical systems (see e.g. [12]):

**Definition 4.2** Let \( \mathcal{M} \) be a normed space, called phase space. A dynamical system describes either the time evolution of a variable \( u(t) \in \mathcal{M} \) by a continuous evolution equation:

\[
\frac{du}{dt} = F(u, t; \mu), \quad \forall t \in (0, \infty),
\]  

(4.1)

or the evolution of a discrete variable \( u^n \in \mathcal{M} \) by a time mapping:

\[
u^{n+1} = F(u^n, n; \mu), \quad \forall n \in \{0, 1, 2, \ldots \},
\]

(4.2)

where \( \mu \) is a control parameter in some parameter space (for example \( \mathbb{R}^m \), \( m \) being an integer), and \( F : \mathcal{M} \to \mathcal{M} \) is a function differentiable with respect to the variable \( u \).

A dynamical system is said to be autonomous when \( F \) does not explicitly depend on \( t \) or \( n \), i.e. \( F = F(u; \mu) \) or \( F = F(u^n; \mu) \). For example, the Navier-Stokes equations (3.8) constitute a dynamical system describing the time evolution of the velocity \( \mathbf{u} \) in the phase space \( \mathbf{J} \), with control parameter the Reynolds number \( Re \). Moreover, they form an autonomous dynamical system if the loading and the Reynolds number does not depend on the time variable \( t \).

Using the semigroup theory of transformations, dynamical systems are alternatively defined by *time evolution (group) operators* \( \psi(\cdot, t) : \mathcal{M} \to \mathcal{M} \) with the properties:

1. \( \psi(\cdot, 0) = id \), where \( id \) is the identity operator in the phase space \( \mathcal{M} \).
Given an initial state $u^0$ at $t^0$, the state $u(t)$ at time $t \geq t^0$ of a dynamical system is thus denoted $u(t) = \psi(u^0, t - t^0)$.

**Definition 4.3** A trajectory, or flow line, evolving from an initial state $u^0$ at $t^0$ under the time evolution operator $\psi$, is defined as the set of points $S_\psi(u^0) \subset \mathcal{M}$ such that:

$$S_\psi(u^0) = \{v \in \mathcal{M}; \ v = \psi(u^0, t - t^0), \ \forall t \in [t^0, \infty)\}.$$ 

Let $\mathcal{V}$ define an open set in $\mathcal{M}$ at time $t^0$. The set $\mathcal{V}$ is transformed under the time evolution operator $\psi$ into the set $\mathcal{V}(t) \subset \mathcal{M}$ at time $t$ such that:

$$\mathcal{V}(t) = \psi(\mathcal{V}, t - t^0) = \{v \in \mathcal{M}; \ v = \psi(u^0, t - t^0), \ \forall u^0 \in \mathcal{V}\}.$$ 

Assuming we know how to measure the “volume” of a set in the phase space $\mathcal{M}$, and letting $\mathcal{V} \subset \mathcal{M}$ be arbitrary, a dynamical system is said to be dissipative when the volume of $\mathcal{V}(t) = \psi(\mathcal{V}, t - t^0)$ is contracted on the time average.

### 4.2. Attractors

In this section, we focus on characterizing the asymptotic behavior of autonomous dissipative dynamical systems. The volume contractions for such systems may happen in two ways: either all the lengths of the volume decrease with time, or only a fraction of the lengths decrease indefinitely while the others increase, but less faster. Although this implies that the deformation can take on various aspects, it is still possible to characterize geometrically the long time behavior of the solutions using the concept of attractors. Note that the definition of attractors is closely related to the long term stability properties of flows.

**Definition 4.4 (Attractor)** Given a dissipative autonomous dynamical system, an attractor is a compact subset $\mathcal{A}$ of the phase space $\mathcal{M}$ which satisfies the following conditions:

1. $\mathcal{A}$ is an invariant set, i.e., $\forall u^0 \in \mathcal{A}$, $\psi(u^0, t - t^0)$ is defined for all $t \geq t^0$, and $\mathcal{A} = \psi(\mathcal{A}, t - t^0)$, for all $t \geq t^0$.

2. There exists an open neighborhood $\mathcal{V} \supset \mathcal{A}$ such that for every $u^0 \in \mathcal{V}$, $\psi(u^0, t - t^0)$ tends to $\mathcal{A}$ as $t \to \infty$. The largest set $\mathcal{V}$ satisfying this property defines the basin of attraction of $\mathcal{A}$.

3. The flow has the recurrence property; trajectories starting from any open subset $\mathcal{V} \subset \mathcal{A}$ intersect $\mathcal{V}$ again and again as $t \to \infty$. 

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The flow has the irreducibility property; the attractor $A$ cannot be decomposed into distinct smaller attractors.

Attractors are viewed as geometric objects in the phase space $\mathcal{M}$ which attract trajectories originated from initial points in the basin of attraction. We next list the classical types of attractors. In the following, we set $t^0 = 0$.

- **Fixed point attractors.** Let $u^0$ be a fixed point in $\mathcal{M}$ such that $u(t) = \psi(u^0, t) = u^0, \forall t > 0$. If all trajectories initiated in a neighborhood of $\{u^0\}$ accumulate at $u^0$ for large $t$, then $\{u^0\}$ is a fixed point attractor.

- **Periodic orbit attractors.** Suppose there exist $u^0 \in \mathcal{M}$ and $T > 0$, such that $u(T) = \psi(u^0, T) = u^0$ and $u(t) = \psi(u^0, t) \neq u^0$ for all $t \in (0, T)$. Then $u^0$ is said to be a periodic point of period $T$ and the set $\{v \in \mathcal{M}; v = \psi(u^0, t), 0 \leq t < T\}$ a periodic orbit. If all trajectories starting from a neighborhood of $\{v\}$ accumulate at $\{v\}$, for large $t$, then $\{v\}$ is a periodic orbit attractor.

- **Quasiperiodic attractors.** Quasiperiodic attractors are a generalization of periodic attractors and are formed by superposition of attracting periodic orbits with incommensurate periods.

These attractors describe manifolds in the phase space $\mathcal{M}$, such as points, circles or tori. One property they all share is that a “small” change in the initial condition remains relatively “small” at all subsequent times. Ruelle and Takens [40] conjectured the existence of a new class of attractors in order to explain the nature of turbulence and its chaotic behavior. These attractors do not share the property above and, therefore, were called strange attractors.

**Definition 4.5 (Strange attractor)** An attractor of a dynamical system $A$ is called a strange attractor if there are trajectories $S_\psi(u^0) \subset A$ which depend sensitively on the initial conditions $u^0$. By sensitive dependence on the initial conditions, one understands that two trajectories, starting from two distinct points $u^0$ and $v^0$ in the attractor $A$, will exponentially depart from each other after a finite time, no matter how close $u^0$ and $v^0$ are.

The new concept of strange attractors have raised great expectations to explain the nature of turbulence. One immediate consequence would be that the chaotic behavior of turbulent flows, although apparently resulting from stochastic laws, could be yet predicted by a fully deterministic model such as the Navier-Stokes equations. The strange attractor theory of turbulence was born (see [30; 22]), and while the existence of strange attractors for the Navier-Stokes equations has not been proven, experimental works on Couette-Taylor flows undertaken by Swinney et al. [4, 5] lead to the observation of what seem to be strange
attractors and would confirm the conjecture advanced by Ruelle and Takens. Nevertheless, the existence of strange attractors has been shown in the case of simple dynamical systems such as the Lorenz equations (see [32, 45]) and the Hénon map (see [2]).

4.3. Characterization of attractors

Given a dynamical system, it is a formidable task to establish the existence of attractors with the possible exception of the fixed point and the periodic attractors (see for example [8] for the Navier-Stokes equations). Several methods have been developed however to experimentally characterize and determine the type of attractor. All consist in computing, over a long time range, a particular solution \( u \) of the evolution equation (4.1) or mapping (4.2) for a given initial condition \( u^0 \), and in extracting out of it a “relevant” time signal \( S(t) \) or time series signal \( S(t^n) \). If the signal is independent of \( t \), the attractor is clearly a fixed point. Otherwise the time signal is postprocessed so as follows.

1. Obtain a power spectrum. The power spectrum is obtained by applying the Fourier transform on the time signal or discrete time series signal. The spectrum is then the graph of the square of the amplitude of the Fourier coefficients versus the frequency or wave number (See [2] for details). If the spectrum shows one fundamental frequency with its harmonics, then the attractor is clearly a periodic orbit. If it contains two or more incommensurate frequencies, we have a quasiperiodic attractor. On the other hand, when the spectrum presents a continuous part (a broad band), it indicates the existence of a strange attractor. The signal is also said aperiodic.

2. Generate time delay reconstruction diagrams. These are obtained by plotting \( S(t) \) versus \( S(t + \tau) \), where \( \tau \) is a given constant. The value of \( \tau \) is usually chosen as the fourth of the period for a periodic signal, so that the time delay reconstruction diagram would represent a perfect circle if the signal \( S(t) \) were the trigonometric function \( \sin(t) \). Time delay reconstruction diagrams have been designed for the large order systems in order to mimic the classical phase portraits used for low order systems.

3. Estimate the dimension \( d_A \) of the attractor in the space \( \mathcal{M} \). There actually exist several types of dimension, namely the Euclidian dimension, the Hausdorff dimension, the fractal dimension or the information dimension. These dimensions coincide for simple attractors. Indeed, the fixed point dimension is \( d_A = 0 \), the periodic orbit dimension \( d_A = 1 \) and the 2-torus dimension \( d_A = 2 \). The dimensions of strange attractors are a priori different and usually take on irrational values.

4. Compute the Lyapunov exponents measuring the rate of divergence of nearby trajectories (see for example [2]). It is also possible to calculate
a dimension based on the Lyapunov exponents according to the conjecture made by Kaplan and Yorke (see Farmer [13]) in order to predict the dimension of the attractor.

We note that the first two methods are generally sufficient to identify the four types of attractors. On the other hand, the last two are valuable to obtain additional information on strange attractors.

4.4. Bifurcations and routes to chaos

A bifurcation refers to the qualitative change in the asymptotic regime of a dynamical system due to a change of the control parameter $\mu$ (see e.g. [53, 44]). In the case of dissipative dynamical systems, a bifurcation translates into a change of the type of attractor as the parameter $\mu$ is varied. One is usually interested in studying the nature of the successive bifurcations when $\mu$ increases. The sequence of bifurcations is usually referred to as the route to chaos since the attractor is usually simple for low values of the parameter and strange (or chaotic) for large values. Researchers have imagined different possible scenarios in an attempt to identify universal routes to chaos in relation with some classes of dynamical systems. Three have retained the attention: namely, the Landau theory of turbulence, the period-doubling cascade [16] and the Ruelle-Takens-Newhouse (RTN) route to chaos [35].

5. On the stability of FE approximations: a numerical study

The Navier-Stokes equations define a dissipative autonomous dynamical system with control parameter $Re$. Here we assume that, given any Reynolds number, there exists an attractor $\mathcal{A}$. The type of the attractor $\mathcal{A}$ is expected to change as $Re$ is increased, but “universal” routes to chaos have not yet been determined. The difficulty is that the nonlinear partial differential equations defined by the Navier-Stokes equations cannot be solved exactly. We appeal to numerical methods to compute approximations, but, at the same time, we should expect some changes in the dynamical properties of these approximations.

When, for instance, the Navier-Stokes equations are discretized in space using the Finite Element method, we obtain the new system of equations (2.4). The partial differential equations are actually transformed into a system of ordinary differential equations, which constitute a new dynamical system of the type (4.1). In addition to the Reynolds number, this dynamical system is controlled by a new parameter, namely the total number of degrees of freedom $N_t$ or simply the size $h$ generated by the finite element spaces $V^h$ and $Q^h$. The attractors associated with this dynamical system, will be denoted $\mathcal{A}_h$. Moreover, when the equations are fully discretized, using the ABCN scheme for instance,
the system of equations (2.5) defines a new dynamical system governed this time by a discrete time mapping of the type (4.2). Such a discretization introduces the timestep $\Delta t$ as a new parameter, and the attractors, when they exist, will be denoted $A_h^{\Delta t}$. The fundamental issues which need to be addressed are then:

1. At a given $Re$, are the attractors $A_h^{\Delta t}$ of the same type when $h$ and $\Delta t$ vary? In other words, do we expect the bifurcations to occur at the same critical Reynolds numbers?

2. Do the routes to chaos, with respect to $Re$, follow the same sequence when $h$ and $\Delta t$ change?

3. Finally, do the attractors $A_h^{\Delta t}$ "converge" (and, if so, in what sense) to the exact attractor $A$ when the parameters $h$ and $\Delta t$ tend to zero?

In [20, 18, 39], numerical simulations of the Navier-Stokes equations were performed with the objectives of showing the existence of strange attractors and of unveiling a possible route to chaos. Our motivation is rather to show how and why the attractors are sensitive to the mesh discretization in order to derive a strategy for the control of the error and stability. Lorenz [33] had this in mind when he conducted some numerical experiments for low-dimensional systems of nonlinear ordinary differential equations. He observed that the time discretized dynamical system led to different attractors depending on the size of $\Delta t$. For very large $\Delta t$, he even failed to obtain attractors. He thus introduced the concept of computational instability and computational chaos.

In the following, we present the results of our numerical experiments performed on two-dimensional channel flows past a cylinder. We have selected this particular case because it is known to undergo the first bifurcation from the steady-state symmetric flow to the periodic vortex shedding flow at a low value of the Reynolds number. The flow domain $\Omega$, shown in Fig. 5.1, is discretized into four meshes of quadrilateral elements, numbered 1 to 4, having respectively 112, 160, 192 and 262 elements. The mesh topology is shown in Fig. 5.2 in the case of mesh 4. In order to break the symmetry of the steady-state flows, we design unsymmetric meshes by placing more elements at the bottom of the cylinder than at the top, which is sufficient to trigger the periodic flows.

The velocity components and the pressure are approximated by piecewise bi-quadratic and piecewise bilinear continuous polynomials. We recall that the Reynolds number is usually defined in this problem as $Re = U_c d / \nu$, where $d$ is the diameter of the cylinder and $U_c$ the inflow velocity at the centerline of the channel. Boundary conditions are prescribed as shown in Fig. 5.1, and, finally, the initial condition $u^0$ is set to zero throughout the domain $\Omega$.

In order to reach a neighborhood of the attractors, long time simulations are performed, keeping in mind that the timestep must remain relatively small in order to satisfy the stability of the ABCN scheme and to minimize the
Figure 5.1: Channel flow past a cylinder: geometry and boundary conditions.

Figure 5.2: Mesh topology.

numerical errors due to the discretization in time. We compromise by defining the time interval as $0 \leq t \leq 500$, where $t$ is the dimensionless time variable, and by fixing the timestep to $\Delta t = 0.02$. We choose nine values of $Re$ ranging from 1 to 300 in the set \{1, 25, 50, 75, 100, 150, 200, 250, 300\}.

In view of characterizing the type of attractors, we extract from our output data time series signals based on the kinetic energy; in particular, at each timestep, we compute the kinetic energy in the “triangular” subdomain $\Omega_s$ of $\Omega$ represented by a dotted line in Fig. 5.1. We then postprocess these time series signals by generating power spectra and time delay reconstruction diagrams. This allows us to classify the attractors into three types; namely, fixed point attractors (.), periodic orbit attractors (O) and other states (X) including quasiperiodic and strange attractors. The attractors we obtain are listed in Table 5.1.

We observe that the attractor $A_{h}^{\Delta t}$ is dependent on the discretization parameter $h$ (with fixed $\Delta t$ and fixed $Re$). In particular, at $Re = 75$, the flow evolves to
Table 5.1: Types of attractor according to the Reynolds number and the spatial discretization: (.) fixed point, (O) periodic orbit with one one fundamental, (X) other, (-) no attractors.

<table>
<thead>
<tr>
<th>Mesh</th>
<th>1</th>
<th>25</th>
<th>50</th>
<th>75</th>
<th>100</th>
<th>150</th>
<th>200</th>
<th>250</th>
<th>300</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>O</td>
<td>X</td>
<td>X</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>O</td>
<td>O</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>O</td>
<td>O</td>
<td>O</td>
<td>O</td>
<td>O</td>
<td>X</td>
</tr>
<tr>
<td>4</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>O</td>
<td>O</td>
<td>O</td>
<td>O</td>
<td>O</td>
<td>X</td>
</tr>
</tbody>
</table>

Figure 5.3: Bifurcation diagrams for the first bifurcation from the steady-state to the periodic-state regime.

a steady-state for mesh 1, and to a periodic-state for the other meshes. On the contrary, at $Re = 200$, the long-term behavior is still periodic for meshes 3 and 4 while it has changed to a more complex state for meshes 1 and 2. Moreover, in the case of mesh 1, no attractors are observed at $Re = 250, 300$ as the solution “blows up” after a finite time.

We further characterize the periodic states by computing amplitudes and frequencies of the kinetic energy signals. The bifurcation diagrams are shown in Figure 5.3. We observe (very qualitatively) that the amplitudes increase more rapidly with respect to the size of the mesh in the range $50 < Re < 75$, but increase more slowly in the interval $75 < Re < 100$. This suggests that the flow stability might be influenced by two phenomena with opposite effects. The amplitudes eventually reach a plateau at high Reynolds numbers for meshes 3 and 4. In the same way, the dimensionless frequencies (or Strouhal numbers) reach a plateau in the case of meshes 3 and 4.

Finally, we show time delay reconstruction diagrams for the mesh 2 in Fig. 5.4 and corresponding power spectra in Fig. 5.5. The diagram exhibits a perfect periodic orbit for $Re = 100$. As the Reynolds number is increased, the curves
Figure 5.4: Time delay reconstruction diagrams for mesh 2.

become more and more complicated, and at $Re = 300$, one may be tempted to conclude that the attractor is strange. However, more investigations are needed to confirm such a result. Similarly, the power spectrum for $Re = 100$ shows one fundamental frequency and its harmonics, indicating a periodic flow as well. Moreover, at $Re = 150$, the power spectrum contains a second frequency, which allows us to conclude that the attractor is quasiperiodic with two incommensurate frequencies. For higher Reynolds numbers, the spectra present almost continuous parts, which would mean that the flow is aperiodic.

These experiments bring to the fore that the long term behavior of finite element approximations of the Navier-Stokes equations are indeed sensitive to the spatial discretization. They also allow us to conclude that the attractor $A_h^\Delta t$ are not necessarily of the same type as $h$ varies and that the bifurcations are shifted with respect to $Re$. This provide a partial answer to the first issue. More experiments are necessary to answer to the other two questions. We have also observed computational instability phenomena introduced by Lorenz [33] with respect to the mesh size. In conclusion, this study emphasizes the need to control the numerical stability of our approximations.
6. Numerical stability analysis

In this section, we attempt to explain the changes in the stability due to the finite element discretization in space. In order to do so, we apply the energy method to finite element approximations of the Navier-Stokes equations. We will illustrate the theoretical results by numerical experiments.

6.1. The energy method applied to FE approximations

We shall consider the finite element approximations \((u_h, p_h)\) as solutions of the semi-discrete Navier-Stokes equations (2.4). Similarly to the continuous case, we introduce the space \(J^h\) of discretely divergence-free functions:

\[
J^h = \{ v \in V^h; \ b(v, q) = 0, \ \forall q \in Q^h \}. \tag{6.1}
\]

Note that we do not necessarily have \(J^h \not\subset J\), nor \(J \not\subset J^h\). Problem (2.4) is recast as finding \(u_h \in L^2(0, T; J^h)\) such that, for all almost \(t \in (0, T)\):

\[
(\partial_t u_h, v) + c(u_h, u_h, v) + Re^{-1} a(u_h, v) = F(v), \ \forall v \in J^h,
\]

\[
u_h = u_h^0, \quad \text{at } t = 0. \tag{6.2}
\]
Let $U_h$ be a steady-state solution at a given Reynolds number. A perturbation $u'_h \in L^2(0, T; J^h)$ to $U_h$ is generated by considering a perturbed initial state $u^0_h$ different to $U_h$ and satisfies the time evolution equation:

$$(\partial_t u'_h, v) + c(U_h, u'_h, v) + c(u'_h, U_h, v) = 0, \quad \forall v \in J^h,$$  

with associated initial condition $u'_h(0) = u^0_h - U_h$ at $t = 0$. The kinetic energy of the perturbation $u'_h$ is then governed by the evolution equation:

$$\frac{1}{2} \frac{d}{dt} \|u'_h\|_0^2 = -c(U_h, u'_h, u'_h) - c(u'_h, U_h, u'_h) - c(u'_h, u'_h, u'_h) - Re^{-1}|u'_h|^2.$$  

(6.3)

Unlike the continuous case, the terms $c(U_h, u'_h, u'_h)$ and $c(u'_h, u'_h, u'_h)$ are maintained in the evolution equation since $u'_h \in J^h$ and $U_h \in J^h$ are not necessarily divergence-free. However, if we consider "small" perturbations and assume that the higher term $c(u'_h, u'_h, u'_h)$ can be neglected with respect to the other terms, then:

$$\frac{1}{2} \frac{d}{dt} \|u'_h\|_0^2 \approx -c(U_h, u'_h, u'_h) - c(u'_h, U_h, u'_h) - Re^{-1}|u'_h|^2.$$  

(6.4)

The kinetic energy of the perturbation $u'_h$ decays if at all times $t > 0$,

$$-c(U_h, u'_h, u'_h) - c(u'_h, U_h, u'_h) - Re^{-1}|u'_h|^2 < 0.$$  

Therefore, we are interested in the following variational problem, where $\mu_h \in \mathbb{R}$:

$$\mu_h = -Re \max_{v \in J^h, |v|_1 = 1} [c(U_h, v, v) + c(v, U_h, v)].$$  

(6.5)

Since $J^h$ is a finite dimensional space, this problem admits a solution $\omega_h \in J^h$. The associated eigenvalue problem reads: find $(\omega_h, \lambda_h) \in J^h \times \mathbb{R}$ such that

$$\frac{1}{2} [c(U_h, \omega_h, v) + c(U_h, v, \omega_h)] +$$  

$$\frac{1}{2} [c(\omega_h, U_h, v) + c(v, U_h, \omega_h)] = \lambda_h Re^{-1} a(\omega_h, v), \quad \forall v \in J^h.$$  

(6.6)

The spectrum of this eigenvalue problem is made of discrete real eigenvalues $\{\lambda_{hi}\}$ such that $\mu_h = \max_i \{\lambda_{hi}\}$. Then the numerical flow $U_h$ is stable if $\mu_h < 1$ and unstable if $\mu_h \geq 1$.

Let $U_h$ be an approximation of an exact steady-state flow $U$. A legitimate question is to ask whether $U_h$ and $U$ exhibit similar flow stability properties. This is essentially answered by comparing the two eigenvalue problems (6.6) and (3.16). We then observe that:
1. The discrete eigenvalue problem is defined on the finite element space $\mathbf{J}^h$ instead of $\mathbf{J}$. The consequences are twofold: The finite element solution is stiffer since $\mathbf{V}^h$ is a smaller space than $\mathbf{V}$. In other words, fewer modes are freed in the discrete problem. On the other hand, the divergence-free constraint is relaxed in the discrete problem since $Q^h \subset Q$. New modes, which are not physical, are then allowed in the numerical solution.

2. The discrete eigenvalue problem contains an extra term since $\mathbf{U}_h$ does not necessarily satisfy the divergence-free constraint. Indeed, we have

$$c(\mathbf{U}_h, \omega_h, \mathbf{v}) + c(\mathbf{U}_h, \mathbf{v}, \omega_h) = -\int_{\Omega} (\omega_h \cdot \mathbf{v}) \nabla \cdot \mathbf{U}_h \, dx,$$

which follows from a property of the trilinear form (see [21]).

Because of these two observations, it becomes apparent why $\mathbf{U}_h$ and $\mathbf{U}$ may have different stability properties. For example, the state $\mathbf{U}_h$ may be stable for all small perturbations in $\mathbf{J}^h$: this means that any numerical flow $\mathbf{u}_h$, close to $\mathbf{U}_h$, eventually evolves towards that state. On the other hand, the state $\mathbf{U}$ may be unstable to some perturbations in $\mathbf{J}$: in that case, there exists a flow $\mathbf{u}$, which, no matter how close approaches the state $\mathbf{U}$, will finally diverge from it and evolve towards another state. This is indeed observed in the previous numerical experiments, as the numerical flow on mesh 1 at $Re = 75$ converges to a steady state whereas the flow on mesh 4 develops into a periodic vortex stretching. At this point, further experiments, taken up in the next section, are needed to evaluate how much each of the finite element spaces $\mathbf{V}^h$ and $Q^h$ affects the flow stability.

### 6.2. Numerical experiments

The objective in this section is to use various discretization spaces $\mathbf{V}^h$ and $Q^h$ in order to study their influence on the flow stability. However, the major difficulty is due to the fact that the spaces $\mathbf{V}^h$ and $Q^h$ are usually related to each other through the discrete LBB condition (2.3). We propose here to take advantage of the $h$-$p$ data structure of our code to partly circumvent this problem: indeed, on a given partition of the domain, we can increase the polynomial orders for the velocity and the pressure independently while preserving the LBB condition. In the following, we will denote $p = (p_u, p_p)$ the pair of polynomial degrees $p_u$ and $p_p$ used for the velocity and the pressure respectively. Starting from the pair $p = (2, 1)$, employed actually in the previous numerical examples, we then consider $p = (3, 1)$ and then $p = (3, 2)$. We note that in the last case, the pressure possesses only edge bubble functions and no interior bubbles in order to satisfy the LBB condition.

We consider again the channel flow past a cylinder investigated earlier. We choose to study the behavior of the solutions on mesh 1 for the Reynolds numbers $Re = 75$, 200 and 250 at which stability discrepancies were observed.
Denoting \( N_p \) and \( N_u \) the number of degrees of freedom for the pressure and each component of the velocity respectively, we shall consider the three cases:

1. \( p = (2, 1) \), for which \( N_u = 500 \) and \( N_p = 138 \),

2. \( p = (3, 1) \), for which \( N_u = 1086 \) and \( N_p = 138 \),

3. \( p = (3, 2) \), for which \( N_u = 1086 \) and \( N_p = 388 \).

We recall here that the objective is not to obtain accurate solutions but to analyze their stability in critical situations. We analyze the behavior of the solutions using the time delay reconstruction diagrams based on the time signal of the kinetic energy in the subdomain \( \Omega_2 \). We also compute the \( L^2(\Omega) \) norm of the divergence of \( \mathbf{u}_h \):

\[
\| \nabla \cdot \mathbf{u}_h \|_0 = \left( \int_\Omega |\nabla \cdot \mathbf{u}_h|^2 \, dx \right)^{1/2}.
\] (6.7)

We will actually evaluate the relative quantity:

\[
q_{\text{div}} = \frac{\| \nabla \cdot \mathbf{u}_h \|_0}{|\mathbf{u}_h|_1}.
\] (6.8)

We display in Fig. 6.1 the results obtained at \( Re = 75 \). In the case \( p = (2, 1) \), the solution evolves towards a steady-state as seen before. When we increase the size of \( \mathbf{V}^h \), taking \( p = (3, 1) \), we observe a periodic state, but with a small frequency \( f = 0.095 \) and a small amplitude \( A = 0.33 \). However, when enriching \( Q^h \) using \( p = (3, 2) \), the new periodic signal has frequency \( f = 0.145 \) and amplitude \( A = 0.58 \); these results compare better with the ones obtained on mesh 4. Moreover, we observe on the plots of \( q_{\text{div}} \) that the transition to the periodic state is delayed in the case \( p = (3, 1) \) when compared to the case \( p = (3, 2) \).

We investigate next the case \( Re = 200 \) and show the time delay reconstruction diagrams in Fig. 6.2. The signal looks chaotic for \( p = (2, 1) \). When we employ the discretization \( p = (3, 1) \), we obtain a periodic signal with \( f = 0.084 \) and \( A = 0.80 \). On the other hand, the periodic signal obtained with \( p = (3, 2) \) has frequency \( f = 0.173 \) and amplitude \( A = 0.53 \). Moreover, comparing the signals provided by the relative quantity \( q_{\text{div}} \), as shown in Fig. 6.3, we observe that the signal for \( p = (3, 1) \) is very irregular although periodic and resembles to the superposition of two signals with commensurate periods. We conclude here that we fail to observe the target periodic regime when using \( p = (3, 1) \) but succeed in the case \( p = (3, 2) \). We also note that \( q_{\text{div}} \) is much smaller in the latter case.

Finally, we recall that the solution at \( Re = 250 \) suddenly blows up at \( t = 200 \) for \( p = (2, 1) \) as shown in Fig. 6.5. Using \( p = (3, 1) \), the finite element solution also
Figure 6.1: Time delay reconstruction diagrams for the solutions on mesh 1 at \( Re = 75 \) for the cases \( p = (2,1) \), \( p = (3,1) \) and \( p = (3,2) \) and evolution of the relative value \( q_{\text{div}} \).

blows up and this at an earlier time. We finally obtained a stable solution when we increase the size of the space \( Q^h \) using \( p = (3,2) \). The solution is indeed periodic with \( f = 0.171 \) and \( A = 0.55 \) and the quantity \( q_{\text{div}} \) is kept much smaller in this case than in the other two discretizations \( p = (2,1) \) and \( p = (3,1) \), as it can be seen in Fig. 6.4.

In conclusion, these experiments have revealed that the solutions have better stability properties when the norm of the divergence of \( u_h \) is kept small. The divergence of \( u_h \) can be viewed as a source of unphysical perturbations which are artificially generated in the numerical flow by the spatial discretization. In consequence, this quantity should be carefully controlled in an adaptive strategy aimed at controlling the numerical stability and numerical error of finite element approximations of the Navier-Stokes equations.

7. Conclusion

In this report, we have briefly reviewed the concept of flow stability in the framework of the classical hydrodynamic stability theory and the more recent
Figure 6.2: Time delay reconstruction diagrams for the solutions on mesh 1 at $Re = 200$ for the cases $p = (2, 1)$, $p = (3, 1)$ and $p = (3, 2)$ and evolution of the relative value $q_{\text{div}}$.

Figure 6.3: Comparison of the signals $q_{\text{div}}$ for $Re = 200$ for $p = (3, 1)$ (left) and for $p = (3, 2)$ (right).

dynamical system theory. We have utilized both theories to investigate the stability properties of numerical approximations of the Navier-Stokes equations. In particular, we have put in evidence the qualitative changes in those properties by performing basic numerical simulations of the channel flow past a
cylinder. Using very coarse meshes to discretize the computational domain, we have shown that the long term behavior of numerical flows was very sensitive to the spatial discretization: for instance, the flow would evolve to a steady-state instead of the expected periodic regime. The main result of this study is that the finite element solutions have better stability properties when the divergence of the discrete velocity is kept small. This has been shown by applying the energy method to the finite element solutions and confirmed with numerical experiments. We have then come to the conclusion that the incompressibility constraint should be carefully enforced if considering an adaptive strategy aimed at controlling the numerical stability and numerical error of approximations of the Navier-Stokes equations.
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